**Module 2: Basic Concepts of Trigonometric Functions**

**IV. Circular Functions and Their Graphs**

After completing this section, you should be able to:

* find trigonometric function values for any real number
* graph the trigonometric functions and analyze their properties

**A. Definitions of Circular Functions**

At the beginning of this module, when we were focusing on the properties of right triangles, trigonometric function values were defined for acute angles. Next, by looking at angles in standard position, as rotations, the definitions of trigonometric functions were extended to apply to any angle. Then angles in radian measure were discussed.

This topic will extend definitions of trigonometric functions again, so that they are defined for all real numbers, rather than just for angles.

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| Take a look at the unit circle.  For an angle *θ* and the associated point  *P*:(*x*, *y*) on the unit circle:  and   That is, the *x*-coordinate of the point is the cosine of the angle, and the *y*-coordinate is the sine of the angle.  *x* = cos *θ* and *y* = sin *θ* |  |

The following table summarizes trigonometric function values for certain special angles, listed in both degree and radian measure.

|  |  |  |  |
| --- | --- | --- | --- |
| **Angle *θ*** | **sin *θ*** | **cos *θ*** | **tan *θ*** |
| 0° or 0 | 0 | 1 | 0 |
| 30° or π/6 | one half | square root of 3 over 2 | square root of 3 over 3 |
| 45° or π/4 | square root of 2 over 2 | square root of 2 over 2 | 1 |
| 60° or π/3 | square root of 3 over 2 | one half | square root of 3 |
| 90° or π/2 | 1 | 0 | undefined |

For every real number *t*, it is possible to assign corresponding values sin *t* and cos *t*.

|  |  |
| --- | --- |
| If *t* is nonnegative, start at the point (1, 0) on the unit circle and mark off a distance *t* along the circle in the positive (counterclockwise) direction.  Label the endpoint of the arc (*x*, *y*). Since the circle has radius 1, the arc of length *t* is equal to the angle rotation *θ* in radians.  That is, *t* = *θ*.  Define sin *t* = sin *θ* = *y* and cos *t* = cos *θ* = *x*. |  |

If *t* is negative, apply the same ideas, but travel in the negative (clockwise) direction.

Because of the link with the unit circle, trigonometric functions of real numbers are often called *circular functions*.

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| **Circular Functions (Trigonometric Functions of Real Numbers)** | |
| |  |  | | --- | --- | | Let *t* be a real number, and let (*x*, *y*) be  the corresponding point on the unit circle. |  | | The trigonometric functions of *t* are defined as follows:   |  |  |  | | --- | --- | --- | | sin *t* = *y* | cos *t* = *x* |  | |  |  |  | | | |  |

**Example IV.A.1:** Given *t* = π/3, what are the coordinates of the corresponding point (*x*, *y*) on the unit circle?

[Solution](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M2-Module_2/popups/Solution-ex4-a1.html)

The first three columns of the table at the beginning of this topic can be transformed to the terminology of circular functions by substituting *t* = *θ*, *x* = cos *θ*, and *y* = sin *θ*, and writing the columns in order by *t*, *x*, and *y*. The transformed table is shown below. The diagram to the right visually displays this information on the unit circle.

|  |  |  |  |
| --- | --- | --- | --- |
| **Real Value** | **Point (*x*, *y*) on the Unit Circle** | |  |
| *t* | *x* = cos *t* | *y* = sin *t* |
| 0 | 1 | 0 |
| π/6 |  | one half |
| π/4 |  |  |
| π/3 | one half |  |
| π/2 | 0 | 1 |

**Example IV.A.2:** Given *t* = 5π/3, what are the coordinates of the corresponding point (*x*, *y*) on the unit circle?

[Solution](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M2-Module_2/popups/Solution-ex4-a2.html)

**Example IV.A.3:** Find sin (–19π/6).

[Solution](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M2-Module_2/popups/Solution-ex4-a3.html)

**Example IV.A.4:** Find sin 5.32, rounded to 4 decimal places.

[Solution](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M2-Module_2/popups/Solution-ex4-a4.html)

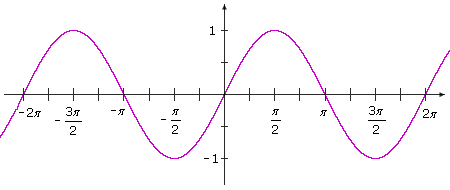
**B. Graphs of the Sine and Cosine Functions**

**The Sine Function**

To get a feel for the graph of the sine function, make a table of values of the sine function and plot the values.

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| |  |  | | --- | --- | | **Input** | **Output** | | *t* | sin *t* | | 0 | 0.0 | | π/6 | 0.5 | | π/4 | square root of 2 by 2 0.7071 | | π/3 | square root of 3 by 2 0.8660 | | π/2 | 1 | | 2π/3 | square root of 3 by 2 0.8660 | | 3π/4 | square root of 2 by 20.7071 | | 5π/6 | 0.5 | | π | 0.0 | | 7π/6 | –0.5 | | 5π/4 | square root of negative 2 by 2–0.7071 | | 4π/3 | square root negative 3 by 2–0.8660 | | 3π/2 | –1 | | 5π/3 | square root negative 3 by 2–0.8660 | | 7π/4 | square root of negative 2 by 2–0.7071 | | 11π/6 | –0.5 | | 2π | 0.0 | | |  |  | | --- | --- | | **Input** | **Output** | | *t* | sin *t* | | 0 | 0.0 | | –π/6 | –0.5 | | –π/4 | square root of negative 2 by 2–0.7071 | | –π/3 | square root negative 3 by 2–0.8660 | | –π/2 | –1 | | –2π/3 | square root negative 3 by 2–0.8660 | | –3π/4 | square root of negative 2 by 2–0.7071 | | –5π/6 | –0.5 | | –π | 0.0 | | –7π/6 | 0.5 | | –5π/4 | square root of 2 by 20.7071 | | –4π/3 | square root of 3 by 20.8660 | | –3π/2 | 1 | | –5π/3 | square root of 3 by 20.8660 | | –7π/4 | square root of 2 by 20.7071 | | –11π/6 | 0.5 | | –2π | 0.0 | |

***y* = sin *t***

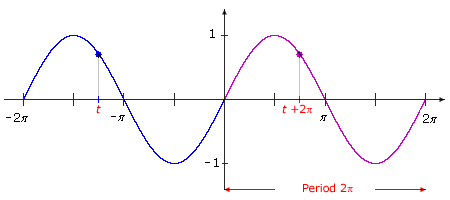


The graph of the sine function has many interesting features.

The graph oscillates between –1 and 1. According to the definition of the sine function, any real number may be input for *t*, so the domain of the sine function is (–∞ , ∞). The range is [–1, 1]. The graph crosses the *t*-axis at each integer multiple of π, so the zeros of the sine function are 0, ±π, ±2π, ±3π, ….

Trace the graph, starting at the origin. When *t* = 0, the *y* value is 0. As *t* increases, the *y* values increase until a peak of 1 is reached when *t* = π/2. Then as t goes from π/2 to 3π/2, the *y* values decrease to –1. As *t* goes from 3π/2 to 2π, the *y* values increase to 0.

If you make a copy of the curve from 0 to 2π and place it on top of the curve from –2π to 0, the two curves coincide. The graph has a pattern that repeats.



For any value of *t*, sin(*t* + 2π) = sin *t*. The sine function is said to be *periodic* with period 2π.

**Periodic Function**

If the graph of a function has a repeating pattern, the function is said to be *periodic*.

More precisely, a function *f* is called *periodic* if there is a positive constant *p* such that  
*f*(*t* + *p*) = *f*(*t*) for all *t* in the domain of *f*. The period *p* of the function is the smallest positive constant for which *f*(*t* + *p*) = *f*(*t*) for all *t* in the domain.

Recall from college algebra that the graph of *y* = *f*(*t* + *p*) is a horizontal shift of the graph of *y* = *f*(*t*). (See [module 1, topic III](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M1-Module_1/S3-Commentary.html#III._Symmetry).)

If you shift the graph of *y* = *f*(*t*) to the left by *p* units, you arrive at the graph of *y* = *f*(*t* + *p*).

Saying that *f*(*t* + *p*) is equal to *f*(*t*) means that if you shift the graph of *f* to the left by *p* units, the graph remains the same.

In the case of the sine function, if you shift the graph of the sine function to the left by 2π, the graph is the same. If you shift the graph of the sine function to the left by less than 2π, the graph does not remain the same. The sine function is periodic with period 2π.

Alternatively, note that *t* and *t* + 2π correspond to coterminal angles, and coterminal angles have the same trigonometric function values. Therefore, sin(*t* + 2π) = sin *t*, and the sine function is periodic with *p* = 2π.

Given a periodic function, there are two measurements of particular interest: how often the graph repeats (the period) and how much the graph oscillates vertically (the amplitude).

**Amplitude**

Given a periodic function having a maximum and a minimum, the **amplitude** is defined to be half of the distance between the maximum and minimum *y* values.

Amplitude = Formula for Amplitude

In the case of the sine function, the amplitude is one half|1 – (–1)| = 1.

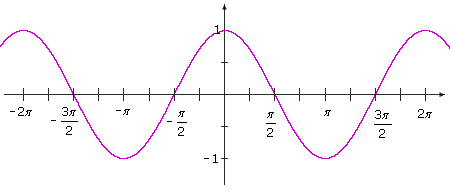
Recall that graphs may exhibit certain types of symmetry. Take a careful look at the graph of the sine function and notice that it is symmetric with respect to the origin. (See [module 1, topic III](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M1-Module_1/S3-Commentary.html#III._Symmetry).) A function *f* whose graph is symmetric with respect to the origin is called an *odd function*, and such a graph satisfies *f*(–*t*) = –*f*(*t*). Therefore, sin(–*t*) = –sin *t*.

**The Cosine Function**

The cosine function can be graphed by using the same approach as the sine function:

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
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| |  |  | | --- | --- | | **Input** | **Output** | | *t* | cos *t* | | 0 | 1.0 | | π/6 | square root of 3 by 20.8660 | | π/4 | square root of 2 by 20.7071 | | π/3 | 0.5 | | π/2 | 0 | | 2π/3 | –0.5 | | 3π/4 | square root of negative 2 by 2–0.7071 | | 5π/6 | square root negative 3 by 2–0.8660 | | π | –1 | | 7π/6 | square root negative 3 by 2–0.8660 | | 5π/4 | square root of negative 2 by 2–0.7071 | | 4π/3 | –0.5 | | 3π/2 | 0 | | 5π/3 | 0.5 | | 7π/4 | square root of 2 by 20.7071 | | 11π/6 | square root of 3 by 20.8660 | | 2π | 1 | | |  |  | | --- | --- | | **Input** | **Output** | | *t* | cos *t* | | 0 | 1.0 | | –π/6 | square root of 3 by 20.8660 | | –π/4 | square root of 2 by 20.7071 | | –π/3 | 0.5 | | –π/2 | 0 | | –2π/3 | –0.5 | | –3π/4 | square root of negative 2 by 2–0.7071 | | –5π/6 | square root negative 3 by 2–0.8660 | | –π | –1 | | –7π/6 | square root negative 3 by 2–0.8660 | | –5π/4 | square root of negative 2 by 2–0.7071 | | –4π/3 | –0.5 | | –3π/2 | 0 | | –5π/3 | 0.5 | | –7π/4 | square root of 2 by 20.7071 | | –11π/6 | square root of 3 by 20.8660 | | –2π | 1 | |

***y* = cos *t***



The graph oscillates between –1 and 1. The domain is (–∞ , ∞) and the range is [–1, 1]. The graph crosses the *t*-axis at Period Range, so the zeros of the cosine function are .

Trace the graph, starting when *t* = 0 and *y* = 1. As *t* increases, the *y* values decrease from the peak of 1 to –1 when *t* = π. Then as *t* goes from π to 2π, the *y* values increase to 1 again.

The cosine function is also periodic with period 2π and has an amplitude of 1. Notice that the cosine graph is symmetric with respect to the *y*-axis. A function *f* whose graph is symmetric with respect to the *y*-axis is called an *even function*, and satisfies *f*(–*t*) = *f*(*t*). Therefore, cos(–*t*) = cos *t*.

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| If you shift the cosine graph to the right by π/2, you arrive at the sine graph.  Therefore, cos(*t* – π/2) = sin *t*. |  |
| Here is a side-by-side comparison of the graphs of sine and cosine: | |

|  |  |
| --- | --- |
| **Sine Function** | **Cosine Function** |
| Sin Function Period Graph | Cosine Function Period Graph |
| Domain: (–∞ , ∞) Range: [–1, 1]  Zeros: 0, ±π, ±2π, ±3π, ...  Periodic with period 2π Amplitude: 1 Odd Function: sin(–*t*) = –sin *t* | Domain: (–∞ , ∞) Range: [–1, 1] Zeros:Period Wave Periodic with period 2π Amplitude: 1 Even Function: cos(–*t*) = cos *t* cos(*t* – π/2) = sin *t* |

**C. Graphs of the Other Trigonometric Functions**

**The Tangent Function**

The domain of a function of the form p(t) over q(t)consists of those inputs *t* in the domains of both *p* and *q* for which *q*(*t*) is nonzero.

Recalling the definitions of the circular functions, since *y* = sin *t*, *x* = cos *t*, and *tan t* = y over xwhen *x*≠ 0, the tangent function is defined as *tan t* =sin t over cos t, when *cos t* ≠ 0.

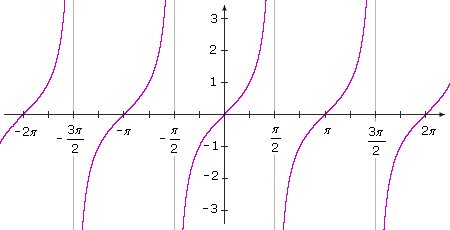
The domain of the sine and cosine functions consists of all real numbers, and by examination of graph of the cosine function, we can see that *cos t* = 0 when  .

Therefore, the domain of the tangent function consists of all real numbers except .

Vertical asymptotes occur at the *t* values for which the denominator *cos t* is zero. (See [module 1, topic II-F](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M1-Module_1/S3-Commentary.html#F._Rational_Functions_and_Asymptotes) for a review of asymptotes.)

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
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| |  |  | | --- | --- | | **Input** | **Output** | | *t* | tan *t* | | 0 | 0 | | π/4 | 1 | | π/2 | undefined | | 3π/4 | –1 | | π | 0 | | 5π/4 | 1 | | 3π/2 | undefined | | 7π/4 | –1 | | 2π | 0 | | |  |  | | --- | --- | | **Input** | **Output** | | *t* | tan *t* | | 0 | 0 | | –π/4 | –1 | | –π/2 | undefined | | –3π/4 | 1 | | –π | 0 | | –5π/4 | –1 | | –3π/2 | undefined | | –7π/4 | 1 | | –2π | 0 | |

***y* = tan *t***



The tangent function has period π. The graph has infinitely many vertical asymptotes. The function has no maximum or minimum, and so the amplitude is not defined. The range is (–∞, ∞).

The graphs of the cotangent, secant, and cosecant functions can be determined using the same approach. Here is a visual comparison:

|  |  |
| --- | --- |
| **Tangent Function** | **Cotangent Function** |
| Tangent Function Graph | Cotangent Function Graph |
| Domain: All real numbers except Period Range  Range: (–∞ , ∞) The graph is periodic with period π. | Domain: All real numbers except ±π, ±2π, ±3π, ...  Range: (–∞ , ∞) The graph is periodic with period π. |
| **Cosecant Function** | **Secant Function** |
| Secant Function Graph  Domain: All real numbers except ±π, ±2π, ±3π, ...  Range: (–∞ , –1] union [1, ∞)  Since the sine is between –1 and 1 and the cosecant is the reciprocal of the sine, the cosecant must be greater than or equal to 1, or less than or equal to –1.  The graph is periodic with period 2π. | https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M2-Module_2/images/Mod2GraphicsFiles/Mod2-SecIV/C-5-sec.gif  Domain: All real numbers except Period Range  Range: (–∞ , –1] union [1, ∞)  Since the cosine is between –1, and 1 and the secant is the reciprocal of the sine, the secant must greater than or equal to 1, or less than or equal to –1.  The graph is periodic with period 2π. |

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